A CANONICAL MODULE CHARACTERIZATION OF SERRE'S (R_1).

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Abstract. In this short note, we give a characterization of domains satisfying Serre's condition (R_1) in terms of their canonical modules. In the special case of toric rings, this generalizes a result of the second author [9] where the normality is described in terms of the "shape" of the canonical module.

1. Introduction

Let A be a noetherian (commutative) domain. Recall that A is said to satisfy Serre's condition (R_1) if all localizations $A_\mathfrak{p}$ at prime ideals $\mathfrak{p}$ of height at most one are regular local rings. In the present note, we characterize this condition in terms of their canonical modules under mild technical conditions (i)–(iii) on A (see §2 for the detail of these conditions). Our main result is the following:

Theorem 2.2. *Let A be a noetherian domain satisfying (i)–(iii) and let $\overline{A}$ denote the integral closure of A. Then following are equivalent:*

(1) The ring A satisfies Serre's (R_1).
(2) There is a canonical module of $\overline{A}$ which is also a canonical module of A.
(3) Some canonical module C of A has an $\overline{A}$-module structure compatible with its A-module structure via the inclusion $A \hookrightarrow \overline{A}$.
(4) For some (actually, any) canonical module C of A, the endomorphism ring $\mathrm{Hom}_A(C, C)$ is isomorphic to $\overline{A}$.

This theorem has a multigraded version. Let R be a $\mathbb{Z}^n$-graded domain, such that R_0 is a field and R is a finitely generated R_0-algebra. Now R admits a *canonical module (i.e., canonical modules in the graded context), which is unique and denoted by ω_R. Similarly, the integral closure $\overline{R}$ of R also admits a *canonical module $\omega_{\overline{R}}$.

Theorem 3.1. *The following are equivalent:*

(1) R satisfies Serre's (R_1).
(2) $\omega_{\overline{R}}$ is a canonical module of R (in the ungraded context).
(3) $\omega_{\overline{R}} \cong \omega_R$ in $^*\mathrm{Mod}\, R$, that is, $\omega_{\overline{R}}$ is a *canonical module of R.

The motivation for these results stems from a related result by the second author for toric rings. Indeed, let $\Bbbk$ be a field and let $\mathbb{M} \subset \mathbb{Z}^d$ be a positive affine monoid, i.e. a finitely generated (additive) submonoid of $\mathbb{Z}^d$ without nontrivial units.

Theorem 1.1 (Theorem 3.1, [9]). *With the above notation, consider the monoid algebra $\Bbbk[\mathbb{M}] = \bigoplus_{a \in \mathbb{M}} \Bbbk x^a$. Then the following are equivalent:*

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(a) $\Bbbk[\mathbb{M}]$ *is normal.*
(b) $\Bbbk[\mathbb{M}]$ *is Cohen-Macaulay and the canonical module* $\omega_{\Bbbk[\mathbb{M}]}$ *is isomorphic to the ideal* $(x^a \mid a \in \mathbb{M} \cap \operatorname{rel-int}(\mathbb{R}_{\geq 0}\mathbb{M}))$ *of* $\Bbbk[\mathbb{M}]$ *as (graded or ungraded)* $\Bbbk[\mathbb{M}]$*-modules.*

Here, for $X \subset \mathbb{R}^d$*,* $\operatorname{rel-int}(X)$ *means the relative interior of* X*.*

Since the ideal $\overline{W}_R := (x^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{Z}\mathbb{M} \cap \operatorname{rel-int}(\mathbb{R}_{\geq 0}\mathbb{M}))$ is known to be a *canonical module of the normalization $\overline{R} = \Bbbk[\mathbb{Z}\mathbb{M} \cap \mathbb{R}_{\geq 0}\mathbb{M}]$, 3.1 yields the following equivalence.

(i) $R = \Bbbk[\mathbb{M}]$ satisfies Serre's (R_1).
(ii) $\overline{W}_R$ is a canonical module of R.

The above fact is clearly analogous to Theorem 1.1, while it uses $\overline{W}_R$ instead of $W_R := (x^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{M} \cap \operatorname{rel-int}(\mathbb{R}_{\geq 0}\mathbb{M}))$. We also obtain a direct generalization of 1.1 as follows.

Theorem 4.2. *Let* $\mathbb{M}$ *be a (not necessarily positive) affine monoid and let* C *be a canonical module of* $R = \Bbbk[\mathbb{M}]$*. Then* R *satisfies Serre's* (R_1) *if and only if there is an injection* $W_R \hookrightarrow C$ *with* $\dim(C/W_R) < d-1$*. Here* d *is the height of the *maximal ideal of* R*.*

2. General context

Definition 2.1. Let $(A, \mathfrak{m}, K)$ be a noetherian local ring of dimension d, and C a finitely generated A-module. We say C is a *canonical module* of A, if we have an isomorphism $\operatorname{Hom}_A(C, E(K)) \cong H^d_{\mathfrak{m}}(A)$, where $E(K)$ is the injective hull of the residue field $K = A/\mathfrak{m}$. If A is not local, we say that a finitely generated A-module C is a canonical module, if the localization $C_{\mathfrak{m}}$ is a canonical module of $A_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of A.

If A is local, a canonical module C is unique up to isomorphism (if it exists). In the general case, it is not necessarily unique. In fact, for a rank one projective module M, $C \otimes_A M$ is a canonical module again.

In this section, unless otherwise specified, A is a noetherian integral domain satisfying the following conditions:

(i) For all maximal ideals $\mathfrak{m}$ of A, we have $\dim A_{\mathfrak{m}} = d$.
(ii) The integral closure $\overline{A}$ is a finitely generated A-module. (A basic reference of this condition is [7, Chapter 12].)
(iii) A admits a dualizing complex $D^\bullet_A$. (In the sequel, $D^\bullet_A$ will mean the *normalized* dualizing complex of A.)

These are mild conditions. In fact, an integral domain which is finitely generated over a field satisfies them. We also remark that $H^{-d}(D^\bullet_A)$ is a canonical module of A. Moreover, if C is a canonical module of A, then the localization $C_{\mathfrak{p}}$ is a canonical module of $A_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ by [1, Corollary 4.3], because A is a domain.

Somewhat surprisingly, the following basic fact does not appear in the literature.

Theorem 2.2. *Let* A *be a noetherian domain satisfying (i)–(iii) and let* $\overline{A}$ *denote the integral closure of* A*. Then following are equivalent:*

(1) The ring A *satisfies Serre's* (R_1)*.*
(2) There is a canonical module of $\overline{A}$ *which is also a canonical module of* A*.*
(3) Some canonical module C *of* A *has an* $\overline{A}$*-module structure compatible with its* A*-module structure via the inclusion* $A \hookrightarrow \overline{A}$*.*

(4) For some (actually, any) canonical module C of A, the endomorphism ring $\mathrm{Hom}_A(C,C)$ is isomorphic to $\overline{A}$.

Proof. (1) $\Rightarrow$ (2) : Assume that A satisfies Serre's (R_1). The short exact sequence

$$0 \to A \to \overline{A} \to \overline{A}/A \to 0$$

yields the following exact sequence:

$$0 = \mathrm{Ext}_A^{-d}(\overline{A}/A, D_A^\bullet) \to \mathrm{Ext}_A^{-d}(\overline{A}, D_A^\bullet) \to \mathrm{Ext}_A^{-d}(A, D_A^\bullet) \to \mathrm{Ext}_A^{-d+1}(\overline{A}/A, D_A^\bullet) = 0.$$

Since A satisfies Serre's (R_1), we have $(\overline{A}/A)_\mathfrak{p} = 0$ for every prime $\mathfrak{p}$ of height one. Hence $\dim \overline{A}/A < d-1$, and the outer terms of the above sequence vanish. To see this, for a maximal ideal $\mathfrak{m}$ of A, note that $\mathrm{Ext}_A^i(\overline{A}/A, D_A^\bullet) \otimes_A A_\mathfrak{m} \cong \mathrm{Ext}_{A_\mathfrak{m}}^i((\overline{A}/A)_\mathfrak{m}, D_{A_\mathfrak{m}}^\bullet)$, where $D_{A_\mathfrak{m}}^\bullet$ is the dualizing complex of the local ring $A_\mathfrak{m}$. Hence we may assume that A is a local ring with the maximal ideal $\mathfrak{m}$. Then, the Matlis dual of $\mathrm{Ext}_A^{-i}(\overline{A}/A, D_A^\bullet)$ is the local cohomology module $H_\mathfrak{m}^i(\overline{A}/A)$.

Anyway, it follows that $\mathrm{Ext}_A^{-d}(\overline{A}, D_A^\bullet) \cong \mathrm{Ext}_A^{-d}(A, D_A^\bullet) \cong H^{-d}(D_A^\bullet)$, and thus $\mathrm{Ext}_A^{-d}(\overline{A}, D_A^\bullet)$ is a canonical module of A. At the same time, since $\overline{A}$ is a finitely generated as an A-module, $\mathrm{Ext}_A^{-d}(\overline{A}, D_A^\bullet)$ is a canonical module of $\overline{A}$.

(2) $\Rightarrow$ (3) : Clear.

(3) $\Rightarrow$ (4) : Let C be a canonical module of A. Since C is a torsionfree A-module of rank 1, it can be regarded as an A-submodule of the field of fractions Q of A, and $\mathrm{Hom}_A(C,C)$ can be identified with $(C:C) := \{\, \alpha \in Q \mid \alpha C \subseteq C \,\}$. Moreover, since C is finitely generated, every $\alpha \in (C:C) \cong \mathrm{Hom}_A(C,C)$ is integral over A (i.e., $\alpha \in \overline{A}$) by the Cayley-Hamilton theorem, hence $(C:C) \subseteq \overline{A}$.

By assumption, there exists a canonical module C' of A which admits an $\overline{A}$-module structure. So for C' it holds that $\overline{A} \subseteq (C':C') \subseteq \overline{A}$. Further, for every maximal ideal $\mathfrak{m}$ of A it holds that $C_\mathfrak{m} = C'_\mathfrak{m}$ and hence $(C_\mathfrak{m} : C_\mathfrak{m}) = (\overline{A})_\mathfrak{m}$. Here $(\overline{A})_\mathfrak{m}$ is the localization of $\overline{A}$ at the multiplicatively closed set $A \setminus \mathfrak{m}$. Thus, it follows that $\mathrm{Hom}_A(C,C) \cong (C:C) = \overline{A}$.

(4) $\Rightarrow$ (1) : Let $\mathfrak{p}$ be a height one prime ideal of A. Since A is a domain, the localization $A_\mathfrak{p}$ is Cohen-Macaulay. Hence we have

$$\overline{A_\mathfrak{p}} = (\overline{A})_\mathfrak{p} \cong [\mathrm{Hom}_A(C,C)]_\mathfrak{p} \cong \mathrm{Hom}_{A_\mathfrak{p}}(C_\mathfrak{p}, C_\mathfrak{p}) \cong A_\mathfrak{p}.$$

This means that A satisfies (R_1). $\square$

In the next result, we assume that A is a noetherian *local* ring satisfying the conditions (ii) and (iii) above. The localization of an integral domain which is finitely generated over a field, and a complete local domain are typical example of these rings. The (S_2)*-ification* of a local ring A s the endomorphism ring $\mathrm{Hom}_A(C,C)$ of its (unique) canonical module C. See [2] for detail.

Corollary 2.3. *Let A be a local ring satisfying the conditions (ii) and (iii) above, and A' its (S_2)-ification. Then A satisfies Serre's (R_1) if and only if so does A' (equivalently, A' is normal).*

Proof. Sufficiency: Clear from the implication (1) $\Rightarrow$ (4) of Theorem 2.2.

Necessity: Take a height 1 prime $\mathfrak{p}$ of A. Let C be a canonical module of A, then $C_\mathfrak{p}$ is a canonical module of $A_\mathfrak{p}$, and $A_\mathfrak{p} \cong \mathrm{Hom}_{A_\mathfrak{p}}(C_\mathfrak{p}, C_\mathfrak{p}) \cong [\mathrm{Hom}_A(C,C)]_\mathfrak{p} = A'_\mathfrak{p}$. If A' satisfies (R_1), then the localization $A'_\mathfrak{p}$ also. $\square$

3. Multigraded context

In this section, let $R = \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} R_{\mathbf{a}}$ be a $\mathbb{Z}^n$-graded domain. We assume that the degree zero part $R_0 = \Bbbk$ is a field and that R is finitely generated as a $\Bbbk$-algebra. Note that the ideal $\mathfrak{m}$ which is generated by all homogeneous non-units is a proper ideal, which contains all proper homogeneous ideals. In other words, R is a **local ring* with **maximal ideal* $\mathfrak{m}$. Note that $\mathfrak{m}$ is not necessarily a maximal ideal in the usual sense, though it is always a prime ideal.

The integral closure $\overline{R}$ of R is also a $\mathbb{Z}^n$-graded *local ring with $\overline{R}_0$ a field. The *maximal ideal $\mathfrak{n}$ of $\overline{R}$ satisfies $\mathfrak{n} = \sqrt{\mathfrak{m}\overline{R}}$ and $\operatorname{ht}\mathfrak{n} = \operatorname{ht}\mathfrak{m}$.

Let $^*\mathrm{Mod}\,R$ be the category of $\mathbb{Z}^n$-graded R-modules, and $^*\mathrm{mod}\,R$ its full subcategory consisting of finitely generated modules. For $M \in {}^*\mathrm{Mod}\,R$, let

$$M^\vee := \operatorname{Hom}_R(M, {}^*E(R/\mathfrak{m})) \cong \bigoplus_{\mathbf{a}\in\mathbb{Z}^n} \operatorname{Hom}_\Bbbk(M_{-\mathbf{a}}, \Bbbk)$$

be the $\mathbb{Z}^n$-graded Matlis dual of M, where $^*E(R/\mathfrak{m})$ is the injective hull of $R/\mathfrak{m}$ in $^*\mathrm{Mod}\,R$. For the information on this duality, see [3, pp.312–313].

A module $C \in {}^*\mathrm{mod}\,R$ is called a **canonical module* of R if it satisfies $C^\vee \cong H^d_\mathfrak{m}(R)$ (cf. [3, §14]), where $d := \operatorname{ht}\mathfrak{m}$. But R_0 is a field, so it holds that $M \cong M^{\vee\vee}$ for all $M \in {}^*\mathrm{mod}\,R$, see [3, p.313]. Hence we can take the Matlis dual of the defining equation and obtain that

$$\omega_R := H^d_\mathfrak{m}(R)^\vee$$

is the unique *canonical module of R.

Theorem 3.1. *The following are equivalent:*

1. *R satisfies Serre's* (R_1).
2. *$\omega_{\overline{R}}$ is a canonical module of R (in the ungraded context).*
3. *$\omega_{\overline{R}} \cong \omega_R$ in $^*\mathrm{Mod}\,R$, that is, $\omega_{\overline{R}}$ is a *canonical module of R.*

Proof. R can be considered as a graded quotient ring of a partial Laurent polynomial ring $S = \Bbbk[x_1^{\pm 1}, \dots, x_l^{\pm 1}, y_1, \dots, y_m]$ admitting a $\mathbb{Z}^n$-grading. So by graded Local Duality [3, Theorem 14.4.1], there is some $\mathbf{a}_0 \in \mathbb{Z}^n$ with $\operatorname{Ext}^c_S(R, S(\mathbf{a}_0)) \cong \omega_R$, where $c = \dim S - \dim R$. Hence ω_R is also a canonical R-module in the ungraded context (i.e., in the sense of Definition 2.1). So the implications (1) $\Leftrightarrow$ (2) $\Leftarrow$ (3) follow directly from 2.2. The implication (1) $\Rightarrow$ (3) can be proved by a similar way to the implication (1) $\Rightarrow$ (2) of 2.2. For this, note that $\operatorname{Ext}^c_S(\overline{R}, S(\mathbf{a}_0)) \cong \omega_{\overline{R}}$ as R-modules. It follows from applying the graded local duality theorem to the R-module $\overline{R}$. □

4. Toric context

In this section, we consider the situation that R is a toric ring, i.e., $R = \Bbbk[\mathbb{M}] = \bigoplus_{a\in\mathbb{M}} \Bbbk x^a$ for some affine monoid $\mathbb{M} \subseteq \mathbb{Z}^n$. Let $\mathcal{C} := \mathbb{R}_{\geq 0}\mathbb{M} \subset \mathbb{R}^n = \mathbb{R} \otimes_\mathbb{Z} \mathbb{Z}^n$ be the polyhedral cone spanned by $\mathbb{M}$, and rel-int$(\mathcal{C})$ its relative interior. Set

$$W_R := (x^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{M} \cap \text{rel-int}(\mathcal{C})),$$

which is a $\mathbb{Z}^n$-graded ideal of R. In [9, Theorem 3.1], the second author showed that, for a Cohen-Macaulay toric ring R, it is normal if and only if W_R is a canonical module. 4.2 below considerably generalizes this result.

Remark 4.1. Set $\overline{\mathbb{M}} := \mathbb{Z}\mathbb{M} \cap \mathcal{C}$. Then $\overline{R} = \Bbbk[\overline{\mathbb{M}}]$ is Cohen-Macaulay and the ideal

$$\overline{W}_R := (x^{\mathbf{a}} \mid \mathbf{a} \in \overline{\mathbb{M}} \cap \text{rel-int}(\mathcal{C}))$$

is its *canonical module $\omega_{\overline{R}}$. So 3.1 specializes to the statement that R satisfies (R_1), if and only if $\overline{W}_R$ is a *canonical module of R, and if and only if $\overline{W}_R$ is a canonical module of R in the ungraded context. The first equivalence is implicitly stated in [8] and the previous work [6] of the first author. The proofs use explicit computation of the local cohomology $H^i_{\mathfrak{m}}(R)$.

Theorem 4.2. *Let $\mathbb{M}$ be a (not necessarily positive) affine monoid and let C be a canonical module of $R = \Bbbk[\mathbb{M}]$. Then R satisfies Serre's (R_1) if and only if there is an injection $W_R \hookrightarrow C$ with $\dim(C/W_R) < d-1$. Here d is the height of the *maximal ideal of R.*

Proof. First, assume that R satisfies (R_1). The canonical inclusion $R \hookrightarrow \overline{R}$ restricts to a homomorphism $W_R \hookrightarrow \overline{W}_R$. Serre's ($R_1$) implies that $\dim \overline{R}/R < d-1$ (cf. the proof of Theorem 2.2) and thus $\dim \overline{W}_R/W_R < d-1$. Moreover, $\overline{W}_R$ is a canonical module of $\overline{R}$, so by Theorem 3.1 it is a canonical module of R as well, so the claim follows.

Next, assume that there is an inclusion $W_R \hookrightarrow C$ with $\dim C/W_R < d-1$. For a prime $\mathfrak{p} \in \operatorname{Spec} R$, we denote by $\mathfrak{p}^*$ the ideal generated by the homogeneous elements in $\mathfrak{p}$. It is known that $\mathfrak{p}^*$ is again a prime ideal and that $R_{\mathfrak{p}}$ is regular if and only if $R_{\mathfrak{p}^*}$ is regular, cf. [5, Propsition 1.2.5].

Consider a prime ideal $\mathfrak{p}$ of height one. If $\mathfrak{p}$ is not homogeneous, then $\mathfrak{p}^* = (0)$ and thus $R_{\mathfrak{p}^*}$ is the field of fractions of R. So $R_{\mathfrak{p}}$ is regular in this case. On the other hand, if $\mathfrak{p}$ is homogeneous then our assumption implies that $C_{\mathfrak{p}} = (W_R)_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$ is a canonical module of $R_{\mathfrak{p}}$, For the second equality we use the fact that

$$W_R = \bigcap \{\mathfrak{p} \in {}^*\operatorname{Spec} R \mid \operatorname{ht} \mathfrak{p} > 0\},$$

where $^*\operatorname{Spec} R$ is the set of $\mathbb{Z}^n$-graded prime ideals of R.

Further $R_{\mathfrak{p}}$ is a one-dimensional domain and thus Cohen-Macaulay, so the injective dimension of the canonical module $\mathfrak{p}R_{\mathfrak{p}}$ is finite, hence $R_{\mathfrak{p}}$ is regular by [4, Corollary 2.3]. It means that R satisfies (R_1). □

Corollary 4.3. *Assume that R satisfies Serre's (S_2). Then the following are equivalent.*

1. *(1) R is normal.*
2. *(2) W_R is a canonical module of R (in the ungraded context).*
3. *(3) $\overline{W}_R$ is a canonical module of R (in the ungraded context).*

Example 4.4. We give two examples to show that Theorem 4.2 and Corollary 4.3 cannot be extended.

(1) Even if R satisfies (R_1), W_R may not be canonical. Indeed, consider the affine monoid

$$\mathbb{M} := \{(a,b) \in \mathbb{N}^2 \mid a+b \equiv 0 \mod 2\} \setminus \{(1,1)\}.$$

It is not difficult to see that W_R has three minimal generators in the degrees $(1,3), (2,2)$ and $(3,1)$. On the other hand, the *canonical module C of $R = \Bbbk[\mathbb{M}]$ has only two generators, in the degrees $(1,1)$ and $(2,2)$. Thus W_R is not a canonical module, even in the ungraded context.

(2) Moreover, even if W_R is canonical, R may not be normal. For example, consider

$$\mathbb{M} := \mathbb{N}^2 \setminus \{(1,0)\}.$$

and $R = \Bbbk[\mathbb{M}]$. Here, $\mathbb{M}$ and thus R are clearly not normal, but nevertheless W_R is a canonical module of R.

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